ON A GENERALIZATION OF SOME INTEGRAL IDENTITIES DUE TO INGHAM AND SIEGEL

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Summary

This paper evaluates generalizations of multi-dimensional integrals due to Ingham and Siegel, and gives a number of applications of these results.

§1. Introduction

At the recent Research Conference on the Theory of Numbers, held at Pasadena, California, we had occasion to discuss some integrals of Ingham and Siegel concerning matrix functions, see [1]. At the conclusion of the talk, it was pointed out by A. Selberg that various generalizations of Siegel's formula existed. In this paper we shall obtain generalizations of both the Ingham and the Siegel identities, following the method given by Ingham in [5].

These formulae will then be applied in two directions. We shall first obtain a generalization of the matrix analogue of Siegel of the scalar Lipschitz identity

$$(1) \qquad \sum_{n=1}^{\infty} n^{s-1} e^{-nx} = \Gamma(s) \sum_{k=-\infty}^{\infty} (x+2\pi i k)^{-s}, Re(s)>1, Re(x)>0.$$

This formula is equivalent to the functional equation for the Riemann zeta function. We surmise that analogous functional equations hold for the generalized zeta-functions we shall define below.

Following this, we shall turn to the problem of evaluating expressions of the form

$$(2) \qquad \begin{vmatrix} \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{12}} & \cdots & \frac{\partial}{\partial x_{1R}} \\ \vdots & & & \\ \frac{\partial}{\partial x_{R1}} & \frac{\partial}{\partial x_{R2}} & \cdots & \frac{\partial}{\partial x_{RR}} \end{vmatrix}^{K} |x_{ij}|^{\ell},$$

where $|x_{ij}|=\det(x_{ij})$, $i,j=1,2,\ldots,N\geq R$ and $x_{ij}=x_{ji}$. Expressions of this type arise in the theory of symmetric functions, in the theory of matric modular functions in the work of H. Maass and in the discussion of stochastic determinants, cf [2].

§2. The Integrals of Ingham and Siegel

The classical integral of Euler reads

$$(1) \qquad \int_0^\infty e^{-xy}x^{s-1}dx=\Gamma(s)y^{-s},\ \mathrm{Re}(s)>0,\ \mathrm{Re}(y)>0.$$

A generalization of this integral, due to Siegel, [7], is

$$(2) \qquad \int_{X>0} e^{-\mathrm{tr}(XY)}|X|^{s-\frac{(n+1)}{2}}dV=\frac{\pi^{\frac{n(n-1)}{4}}\Gamma(s)\Gamma(s-1/2)\ldots\Gamma(s-\frac{n-1}{2})}{|Y|^{s}}$$

Here X is a symmetric matrix $X=(x_{ij})$ and Y is positive definite. The symbol $|X|$ represents the determinant of X, $dV=\prod_{i\leq j} dx_{ij}$, and the integration is over the region where X is positive definite. The real part of s is taken to be positive and sufficiently large. From the right-hand side we see that $\mathrm{Re}(s)>\frac{n-1}{2}$, where n is the dimension of X, is sufficient.

An evaluation of related classes of integrals is given by Bochner, [4]. Analogues of the Beta integral also exist, cf. Siegel, [7], p.42. These integrals arise in connection with Siegel's theory of matric modular functions.

Independently, in connection with some problems in multivariate analysis of Wishart and Bartlett, Ingham demonstrated

the equality

$$(3) \qquad \left(\frac{1}{2\pi i}\right)^{p(p+1)/2} \int \cdots \int e^{tr(CS)} |S|^{-k} \prod_{i\leq j} ds_{ij}$$

$$= \frac{|C|^{k-1/2p-1/2}}{(2\sqrt{\pi})^{\frac{p(p-1)}{2}} \Gamma(k)\, \Gamma(k-1/2) \ldots \Gamma(k-1/2p+1/2)} \quad \text{, if } C$$

is positive definite

$= 0$ otherwise.

This is an extension of the familiar formula

$$(4) \qquad \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{cs} s^{-k} ds = \frac{c^{k-1}}{\Gamma(k)} \quad , \ c > o$$

$= 0$, otherwise,

where $a > o$, $k > 1$.

In (3) the integration with respect to $s_{k\ell}$ is along the line $a_{k\ell} + it_{k\ell}$ where $-\infty < t_{k\ell} < \infty$ and $A = (a_{k\ell})$ is taken to be positive definite. The parameter k is taken initially to be sufficiently large so that the integral converges absolutely.

It is sufficient to establish one or the other of these integrals, since an application of the Laplace inversion formula derives either from the other. We shall restrict ourselves,

therefore, to deriving a generalization of Ingham's formula.

§3. The Generalized Ingham Formula

Let S be a symmetric matrix of order p, and write

(1) $$S_k = (s_{ij}),\ 1 \le i,j \le k.$$

We shall use the notation $|S_k|$ to denote the determinant of S_k. The result we wish to establish is

(2) $$\left(\frac{1}{2\pi i}\right)^{p(p+1)/2} \int \cdots \int e^{tr(CS)} |S_p|^{-k_p} |S_{p-1}|^{-k_{p-1}} \cdots |S_1|^{-k_1} \pi ds_{ij} \atop {i \le j}$$

$$= \frac{(2\sqrt{\pi})^{\frac{-p(p-1)}{2}} |C^{(1)}|^{\sum_{i=1}^{p} k_i - \frac{(p+1)}{2}} |C^{(2)}|^{-k_1} \cdots |C^{(p)}|^{-k_{p-1}}}{\Gamma(k_p)\Gamma(k_p+k_{p-1}-1/2)\cdots\Gamma(\sum_{i=1}^{p} k_i - \frac{(p+1)}{2})}$$

if C is positive definite,

= 0, otherwise.

Here the integration is taken over the same type of region as before. The parameters $k_1, k_2, \ldots, k_{p-1}$ are to be chosen so that all the expressions $k_p, k_p+k_{p-1}-1/2, \ldots, \sum_{i=1}^{p} k_i - \frac{p}{2} + 1/2$ are positive. For this it is sufficient that k_p be sufficiently large, if the k_i are arbitrary.

The matrices $C^{(k)}$, $k=1,2,\ldots p$ are defined as follows

(3) $$C^{(k)} = (c_{ij}),\ i,j=k,\ldots,p.$$

§4. Proof

The method we employ is precisely that used by Ingham [5], in the case where $k_1=\ldots=k_{p-1}=0$, and depends upon an induction over p, starting with the known case $p=1$.

Denote the variables $s_{1p}, s_{2p},\ldots,s_{p-1,p}, s_{pp}$ by $v_1, v_2,\ldots, v_{p-1}$ and u respectively and the parameters $c_{1p}, c_{2p},\ldots,c_{p-1,p}, c_{pp}$ by $c_1, c_2,\ldots,c_{p-1}$, f respectively,

we may write

(1) $$|S_p| = u|S_{p-1}| + \begin{vmatrix} s_{11} & s_{12} & \cdots & s_{1,p-1} & v_1 \\ \vdots & & & & \\ s_{p-1,1} & \cdots & & s_{p-1,p-1} & v_{p-1} \\ v_1 & \cdots & & v_{p-1} & 0 \end{vmatrix}$$

$$= |S_{p-1}|\,u - S'_{p-1}(v,v),$$

where $S'_{p-1}(v,v)$ is the quadratic form obtained using the adjoint matrix to S'_{p-1}. This yields

(2) $$|S_p| = |S_{p-1}|\,(u - S^{-1}_{p-1}\,(v,v)),$$

where $S_{p-1}^{-1}(v,v)$ is the quadratic form obtained using the inverse matrix to S_{p-1}.

The integrand in (3.2) may be written

$$(3)\qquad e^{tr\,(CS_{p-1})}\, e^{2\sum_{k-1}^{p-1} c_k v_k}\, e^{\rho u}\, |u-S_{p-1}^{-1}(v,v)|^{-k_n}$$

$$|S_{p-1}|^{-(k_n+k_{n-1})}\, |S_{p-2}|^{-k_{n-2}} \dots |S_1|^{-k_1}.$$

We may integrate with respect to $u=s_{pp}$, keeping the other variables fixed. Let $u-S_{p-1}^{-1}(v,v)=w$. Then, with $\beta > 0$,

$$(4)\qquad \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \exp\,[\rho w+\rho S_{p-1}^{-1}(v,v)]\; w^{-k_n}\, dw$$

$$= \exp\,[\rho S_{p-1}^{-1}(v,v)]\; \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{\rho w} w^{-k_n}\, dw$$

Since $\rho>0$, (since $\rho=c_{pp}$,) this yields as the remaining integrand

$$(5)\qquad \rho^{k_{p-1}}\, e^{tr(CS_{p-1})}\, \exp\,[\rho S_{p-1}^{-1}(v,v)+2\sum_{k=1}^{p-1} c_k v_k]$$

$$|S_{p-1}|^{-(k_p+k_{p-1})}\quad |S_{p-2}|^{-k_{p-2}} \dots |S_1|^{-k_1},$$

Since $f > 0$ and $S_{p-1}^{-1}(v,v)$ is positive definite for real v_k, we may integrate with respect to the variables $v_1, v_2, \ldots, v_{p-1}$. Applying the well-known result

$$(6) \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(u,Bu) + 2 \sum_{k=1}^{p-1} c_k u_k} \prod_1 du_1$$

$$= \pi^{p/2} |B|^{-1/2} e^{(c,Bc)},$$

for B positive definite, we obtain as a result of integrating with respect to the $p-1$ variables s_{kp}, $k=1,2,\ldots,p-1$, the new integrand

$$(7) \qquad f^{k_n - 1/2p - 1/2} \exp \left(\sum_{k,l=1}^{p-1} b_{kl} s_{kl} \right)$$

$$|S_{p-1}|^{-(k_p + k_{p-1}) + 1/2} |S_{p-2}|^{-k_{p-2}} \ldots |S_1|^{-k_1},$$

where

(8) $$b_{k\ell} = c_{k\ell} - \frac{c_{kp}c_{\ell p}}{c_{pp}}, \quad k,\ell=1,2,\ldots,p-1.$$

The remainder of the proof is inductive, with the last step being the evaluation of the determinants $|B^{(k)}|$ formed from $B=(b_{k\ell})$, in terms of the determinants $|C^{(k)}|$ as defined in (3.3).

§5. Some Determinants

In order to see how to obtain the general result, consider the 3x3 determinant

(1) $$D_3 = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

with $c_{ij} = c_{ji}$.

Multiply the 3rd row by c_{13}/c_{33} and subtract from the first row; multiply the 3rd row by c_{23}/c_{33} and subtract from the second row. The result is

(2) $$D_3 = \begin{vmatrix} c_{11} - \dfrac{c_{13}c_{31}}{c_{33}} & c_{12} - \dfrac{c_{13}c_{32}}{c_{33}} & 0 \\ c_{21} - \dfrac{c_{23}c_{31}}{c_{33}} & c_{22} - \dfrac{c_{23}c_{32}}{c_{33}} & 0 \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = c_{33}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= c_{33}|B^{(1)}|$$

Proceeding in the same way in the general case we see that

$$(3) \qquad |B^{(k)}| = |C^{(k)}|/c_{pp}.$$

With this result established, the inductive proof of the formula in (3.2) proceeds easily, starting with the case p=1.

§6. An Extension of the Lipschitz-Siegel Identity

Combining the evaluation of the integral in (2.2) with the Poisson summation formula, Siegel established the following identity

$$(1) \qquad \sum_{X>0} |X|^{\rho-\frac{(n+1)}{2}} e^{-tr(XY)} = \beta \sum_{K} |Y+2\pi i K|^{-\rho},$$

for the real part of Y positive definite. Here X is an nxn positive definite matrix, $|X|$ is the determinant of X, and the summation on the left is over all positive definite integer matrices. On the left the summation is over all symmetric semi-integers, that is matrices whose main diagonals are integral and whose elements off the main diagonal are halves of integers. The constant β is given by

$$(2) \qquad \beta = \pi^{\frac{n(n-1)}{4}} \Gamma(\rho)\,\Gamma(\rho-1/2)\ldots\Gamma(\rho-(\tfrac{n-1}{2}))$$

The parameter ρ is taken to be sufficiently large.

The generalization of Siegel's integral obtained from (3.2) is

$$(3) \qquad \int_{X>0} \frac{|X|^{\sum_{i=1}^{p} k_i - \frac{(p+1)}{2}} e^{-tr(XY)}}{|X^{(2)}|^{k_1}|X^{(3)}|^{k_2}\cdots|X^{(p)}|^{k_{p-1}}} \prod_{i\leq j} dx_{ij}$$

$$= \frac{(\sqrt{\pi}\,)^{\frac{p(p-1)}{2}}\,\Gamma(k_p)\,\Gamma(k_p+k_{p-1}-1/2)\ldots\Gamma(\sum_{i=1}^{p} k_i - \frac{p}{2} + 1/2)}{|Y_p|^{k_p}|Y_{p-1}|^{k_{p-1}}\cdots|Y_1|^{k_1}},$$

where

$$(4) \qquad |X^{(k)}| = |x_{ij}|,\ i,j=k,\ldots,p,$$

$$|Y_k| = |y_{ij}|,\ i,j=1,\ldots,k.$$

The restriction in the k_i is that each of the expressions $k_p+k_{p-1}+\ldots+k_R - \frac{R}{2}$ be positive.

Applying the Poisson summation formula we obtain the following generalization of (1).

$$(5) \qquad \sum_{X>0} \frac{|X^{(1)}|^{\sum_{i=1}^{p} k_i - \frac{(p+1)}{2}} e^{-tr(XY)}}{|X^{(2)}|^{k_1} |X^{(3)}|^{k_2} \cdots |X^{(p)}|^{k_{p-1}}}$$

$$= \varrho \sum_{K} |(Y+2\pi i K)_{pp}|^{-k_p} |(Y+2\pi i K_{(p-1)}|^{-k_{p-1}} \cdots |(Y+2\pi i K)_1|^{-k_1},$$

where the sum is over all symmetric half-integers, and ϱ is the constant occurring in (3). The series will converge for k_p sufficiently large compared to the other k_i.

§7. Generalized Zeta-Functions

The zeta-function

$$(1) \qquad \zeta_n(s) = \sum_{\{X\}} |X|^{-s},$$

where the summation is over a reduced set of positive definite integer matrices has been considered by Maass, [6], and a functional equation derived for the case of (2x2)-matrices, cf also, [3].

Since the existence of a Lipschitz identity is equivalent to a functional equation for the zeta-function, we surmise that a corresponding functional equation holds for the generalized

zeta-function

(2) $$\mathfrak{F}_n(x_1,s_2,\ldots,s_n) = \sum_{\{X\}} |X_n|^{-s} |X_{n-1}|^{-s_2-1} \ldots |X_1|^{-s_n}$$

This we shall discuss in a subsequent paper.

§8. Generalized Eisenstein Series

Just as matric modular functions are usually formed by means of Eisenstein series of the form

(1) $$f_n(X,s) = \sum_{\{K,L\}} |KX+L|^{-s},$$

where the summation is over a suitably reduced set of K and L, so we can form generalizations of these series having the form

(2) $$f_n(X,s_1,s_2,\ldots,s_n) = \sum_{\{K,L\}} |(KX+L)_n|^{-s_1} |(KX+L)_{n-1}|^{-s_2} \ldots |(KX+L)_1|^{-s_n}$$

We shall discuss these series in more detail subsequently.

§9. Derivatives of Determinants

Let us now consider the problem of determining the result of applying the operator

(1) $$O_K = \begin{vmatrix} \frac{\partial}{\partial c_{11}} & \frac{\partial}{\partial c_{12}} & \cdots & \frac{\partial}{\partial c_{1K}} \\ \vdots & & & \\ \frac{\partial}{\partial c_{K1}} & \frac{\partial}{\partial c_{K2}} & \cdots & \frac{\partial}{\partial c_{KK}} \end{vmatrix}$$

to a power product of the form $|C^{(1)}|^{a_1}|C^{(2)}|^{a_2}\dots|C^{(p)}|^{a_p}$.

The key to the results we shall obtain is the observation that

$$(2)\qquad O_K\, e^{\mathrm{tr}(CS)} = |S_K|\, e^{\mathrm{tr}(CS)}$$

Consequently, applying O_K to both sides of (3.2), we obtain an immediate evaluation of O_K applied to a product of the form

$|C^{(1)}|^{\sum_{i=1}^{p} k_i - \left(\frac{p+1}{2}\right)}|C^{(2)}|^{-k_1}|C^{(3)}|^{-k_2}\dots|C^{(p)}|^{-k_{p-1}}$. Since $k_1, k_2, \dots, k_{p-1}$ may be arbitrary, positive or negative, provided that k_p is large enough, we obtain the result for arbitrary $a_2, a_3, \dots, a_k$ above, provided that a_1 is large enough. The result obtained in this case extends by analytic continuation to all other values.

In particular, we note that

$$(3)\qquad O_K^R\, |C^{(1)}|^N = \alpha_{R,N}\, |C^{(1)}|^{N-R}\, |C^{(K+1)}|^R ,$$

where $\alpha_{R,N}$ may be determined explicitly as a quotient of gamma functions.

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